HEAT PERTURBATION SPECTRUM OF INCOMPRESSIBLE FLUID FLOWS

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The problem of behavior of thermal (or concentration) perturbations in a fluid flow is of considerable interest in the theory of heat and mass transfer, and in various other applications (see, for example, [1 to 3]). In the case of a steady state flow any temperature (or admixture concentration) perturbation may be considered as a superposition of "normal" perturbations, exponentially dependent on time, on the characteristic decrements. The problem is thus reduced, in essence, to the determination of the spectrum of normal characteristic perturbations. The thermal perturbation spectrum of certain incompressible fluid flows between two isothermal planes is analyzed below.

1. If there are no internal heat sources in the fluid, then the temperature perturbations are defined by the heat conductivity Eqs.

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi \Delta T \tag{1.1}$$

where the thermal diffusivity coefficient χ is assumed to be constant. Thermal perturbations are considered to be so small, that any free convection effects may be disregarded. Velocity $\mathbf{v}$ is assumed to be known.

We shall consider a flow of fluid between two parallel planes $x = \pm h$. Selection of the z-axis in the direction of flow results in there being only one velocity component z different from zero, with $v_x = v_y = 0$, $v_z = v(x)$. We rewrite Eq. (1.1) in a dimensionless form, selecting as the velocity unit the characteristic flow velocity U, and as units of length and time h and h^2/χ respectively, and using an arbitrary unit for temperature

$$\frac{\partial T}{\partial t} + pv(x)\frac{\partial T}{\partial z} = \Delta T, \qquad p = \frac{Uh}{\chi} \tag{1.2}$$

Here all values are dimensionless, and p is the Péclet number. The normal temperature perturbations have the form

$$T(x, z, t) = \theta(x)\exp(-\lambda t + ikz) \tag{1.3}$$

where $\theta(x)$ is the perturbation amplitude, k the real wave number, and λ the perturbation decrement. Substituting (1.3) into (1.2), we obtain the amplitude Eq.

$$\theta'' + [\lambda - k^2 - ikpv(x)]\,\theta = 0 \tag{1.4}$$

As at the isothermal planes $x = \pm 1$, the perturbations vanish

$$\theta(\pm 1) = 0 \tag{1.5}$$

The boundary value problem (1.4), (1.5) defines the characteristic perturbation spectrum $\theta(x)$ together with corresponding decrements λ. If $\lambda = \lambda_r + i\lambda_i$, then its real part λ_r represents the decrement itself, while its imaginary part λ_i defines the perturbation frequency and phase velocity. It is obvious that these assumptions rule out the presence of any mechanisms (such as "Superheating"), which might lead to instability, and show that perturbations must decay. This can be directly ascertained from Eq. (1.4). In order to prove this, we multiply this equation by the complex conjugate solution $\theta^*(x)$, integrate with respect to x from -1 to $+1$, and add the obtained integral relationship to the complex conjugate. We then obtain

$$(\lambda + \lambda^*) \int_{-1}^{1} |\theta|^2 dx = 2 \int_{-1}^{1} (|\theta'|^2 + k^2 |\theta|^2)\, dx \tag{1.6}$$

It is evident from this that $\lambda + \lambda^* = 2\lambda_r > 0$, which proves that normal perturbations are alway damped.

It can be readily shown in the same way that normal thermal perturbations decay in an incompressible fluid moving in an arbitrary closed cavity with isothermal boundaries (such a motion may, for example be generated by motions of a part of the boundary).

2. The spectrum analysis at low velocities of flow may be carried out in a manner similar to that used in the analysis of hydrodynamic perturbations in a plane-parallel flow [4].

We expand the amplitude and the decrement into a power series of parameter ikp (essentially of flow velocity powers)

$$\theta = \theta^{(0)} + (ikp)\, \theta^{(1)} + (ikp)^2 \theta^{(2)} + \ldots \tag{2.1}$$

$$\lambda - k^2 = \mu^{(0)} + (ikp)\, \mu^{(1)} + (ikp)^2 \mu^{(2)} + \ldots \tag{2.2}$$

Equations of successive approximations are

$$\theta^{(0)''} + \mu^{(0)}\theta^{(0)} = 0, \qquad \theta^{(1)''} + \mu^{(0)}\theta^{(1)} = -\mu^{(1)}\theta^{(0)} + v\theta^{(0)} \tag{2.3}$$

$$\theta^{(2)''} + \mu^{(0)}\theta^{(2)} = -\mu^{(1)}\theta^{(1)} - \mu^{(2)}\theta^{(0)} + v\theta^{(1)}$$

. . . .

with boundary conditions for amplitudes $\theta^{(k)}$

$$\theta^{(k)}(\pm 1) = 0 \tag{2.4}$$

The first of Eqs. (2.3) defines the perturbation spectrum in a fluid at rest ($p = 0$)

$$\mu_n^{(0)} = {}^1/_4\, (n+1)^2 \pi^2 \qquad (n = 0,\ 1,\ 2, \ldots) \tag{2.5}$$

$$\theta_n^{(0)} = \cos {}^1/_2\, (n+1)\, \pi x \qquad (n = 0,\ 2,\ 4, \ldots)$$

$$\theta_n^{(0)} = \sin {}^1/_2\, (n+1)\, \pi x \qquad (n = 1, 3, 5, \ldots) \tag{2.6}$$

The amplitudes of zero order approximation $\theta_n^{(0)}$ are orthogonal and normalized

$$\int_{-1}^{1} \theta_n^{(0)} \theta_k^{(0)}\, dx = \delta_{nk} \tag{2.7}$$

For the first and higher order approximations we obtain nonhomogenous equations, from the condition of solvability of which we derive corrections for decrements $\mu_n^{(1)}$, $\mu_n^{(2)}$,... Amplitudes $\theta_n^{(1)}$, $\theta_n^{(2)}$,... may be sought in the form of expansions of eigenfunctions of the zero order approximation (2.6)

$$\theta_n^{(k)} = \sum_m a_{nm}^{(k)} \theta_m^{(0)} \tag{2.8}$$

Expansion coefficients $a_{nm}^{(k)}$ are found in the usual manner from the successive approximation Eqs. (2.3), while from the solvability conditions we obtain

$$\mu_n^{(1)} = V_{nn}, \quad \mu_n^{(2)} = \sum_{m \neq n} \frac{V_{mn}^{\ 2}}{\mu_n^{(0)} - \mu_m^{(0)}}, \ldots \qquad \left(V_{mn} = \int_{-1}^{1} \theta_m^{(0)} v \theta_n^{(0)}\, dx\right) \tag{2.9}$$

3. When the flow velocity profile is of even parity $v(x) = v(-x)$, then obviously two types of perturbation are possible with respect to x, namely even and odd. Decrement corrections, beginning with decrement $\mu_n^{(1)}$, generally differ in this case from zero. This means that thermal perturbations in the presence of an even velocity profile have for an arbitrary small p the character of thermal waves, spreading along the stream with a certain phase velocity defined by $\mu_n^{(1)}$.

As an example we shall adduce the first two decrement corrections for the case of a Poiseuille flow $v = 1 - x^2$(*)

$$\mu_n^{(1)} = \frac{2}{3} + \frac{2}{\pi^2 (n+1)^2}, \qquad \mu_n^{(2)} = \left(\frac{4}{\pi}\right)^6 \sum_{m \neq n} \frac{(n+1)^2 (m+1)^2}{[(n+1)^2 - (m+1)^2]^5} \tag{3.1}$$

(*) M. Oreshina and Z. Shtarkman had participated in the computations.

Summation in the quadratic correction formula for $\mu_n^{(2)}$ is carried out with respect to indices m of the same parity as that of the level number n.

The values of $\mu_n^{(1)}$ and $\mu_n^{(2)}$ for low levels of the spectrum are shown in the Table.

n	$\mu_n^{(1)}$	$\mu_n^{(2)}$
0	0.8693	—0.001185
1	0.7173	—0.001116
2	0.6892	0.0002357
3	0.6793	0.0003099
4	0.6748	0.0002545
5	0.6723	0.0001988
6	0.6708	0.0001559
7	0.6698	0.0001244

With known $\mu_n^{(1)}$ and $\mu_n^{(2)}$ the real and imaginary parts of decrements may be derived for low velocity flows from expansion (2.2)

$$\lambda_{nr} = k^2 + \mu_n^{(0)} - k^2p^2\mu_n^{(2)} + \ldots$$

$$\lambda_{ni} = kp\mu_n^{(1)} + \ldots \tag{3.2}$$

Parameter $\mu_n^{(1)}$ defines (at small values of p) the thermal perturbation phase velocity in terms of units of the flow maximum velocity

$$c = \frac{\lambda_i}{kp} = \mu_n^{(1)} \tag{3.3}$$

The thermal perturbation phase velocity, as may be seen from the Table and (3.1), is independent of the wave number k (there is no dispersion). From (3.1) we deduce that for large n we have $c = 2/8$, which means that the phase velocity of small scale perturbations coincides with the mean flow velocity.

Values of $\mu_n^{(2)}$ are quite small. For example, with $k = 1$, the values of λ_r practically do not differ from values of damping increments in a fluid at rest up to $p \sim 10$.

4. It will be readily seen that in the case of an odd velocity profile with $v(x) = -v(-x)$, all negative decrement corrections vanish

$$\mu^{(1)} = \mu^{(3)} = \ldots = 0 \tag{4.1}$$

Expansion (2.2) in this case becomes

$$\lambda = k^2 + \mu^{(0)} - k^2p^2\mu^{(2)} + \ldots \tag{4.2}$$

Thus, at low velocities the decrements are real, frequencies and phase velocities are zero, and the corresponding perturbations decay monotonously. Obviously this result cannot be extrapolated, to finite values of p, as long as there may exist (and in fact it does exist) on the p-axis a singular point up to which expansions into power series of p are valid. In order to elucidate the character of this singularity it is necessary to consider the intersection of real decrements λ_n. The position here is exactly the same as in the case of hydrodynamic perturbations in two-dimensional flows having an odd velocity profile [4]. Repeating the reasoning given in paper [4], we can readily ascertain that with a decrement spectrum "simple" intersections, corresponding to an ordinary degeneration, are not possible, and that for specific values of p only 'mergers' of two adjacent real levels of different parities, with the formation of complex conjugate decrements are possible (formation of oscillatory perturbations with equal but opposite phase velocities).

The Galerkin method may be used in this case for spectrum determination, with the "unperturbed" amplitudes $\theta_n^{(0)}$, defined by (2.6) selected as basic functions. For finite values of p, the amplitude approximation is expressed by

$$\theta = \sum_{n=0}^{N-1} c_n\theta_n^{(0)} \tag{4.3}$$

From the orthogonality condition of the Galerkin method we obtain for coefficients c_n the system of N homogenous linear Eqs.

$$\sum_{n=0}^{N-1} c_n\,[(\lambda - \lambda_n^{(0)})\,\delta_{mn} - ikpV_{mn}] = 0 \qquad (m = 0, 1, 2, \ldots, N-1) \tag{4.4}$$

Here V_{mn} are matrix elements defined in (2.9). In the case of an odd profile $v(x)$ these matrix elements differ from zero for indices of different parity only. For specific velocity profiles the eigenvalues λ of the matrix of system (4.4) define the characteristic decrements in terms of parameters p and k. Computations were carried out for a Couette flow $v = x$, and for a flow having a cubic velocity profile $v = x - x^3$. Approximation (4.3) contained $N = 18$ functions. Matrix diagonalization was carried out on an 'Aragats' electronic computer. Lower levels of the decrement spectrum are shown on the Fig. 1 for the wave number $k = 1$.

Real parts of decrements λ_r for the eight lower levels, and the second powers of phase

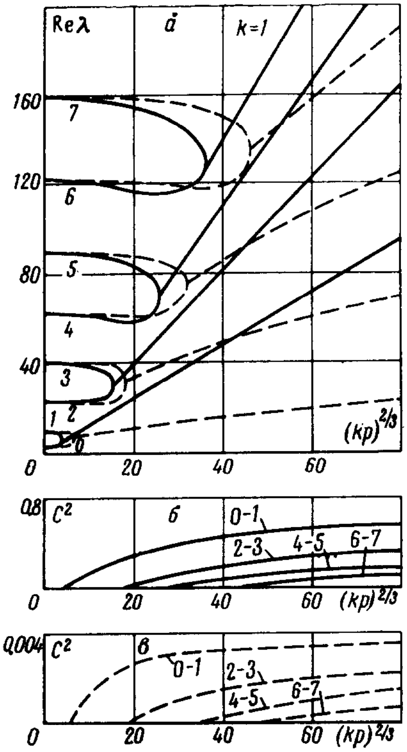

Fig. 1

velocity c^2 (in units of flow velocity) are represented in terms of the Peclet number p. Solid lines relate to the Couette flow, and the dotted lines to the cubic velocity profile flow. The pairwise merging of real levels with formation of complex conjugate pairs is seen to occur with increasing p. Oscillatory perturbations (thermal waves) appear at a finite value of $p = p_*$, while the perturbation phase velocity becomes zero at the point of merger and increases with increasing p. Oscillatory perturbations occur as the result of the two lower levels being merged when $p_* \sim 10$.

5. In concluding, we shall consider the thermal perturbation spectrum in a uniform transverse flow between permeable planes. We assume that in the plane $x = -1$ we have a uniformly incoming flow which is uniformly sucked away at the same rate at plane $x = 1$. In this case the velocity of the fluid flow between the two planes has one component only which is perpendicular to the planes, and is independent of the coordinates. For the normal perturbation amplitude we now have instead of (1.4) an Eq. with constant coefficients

$$\theta'' - p\theta' + (\lambda - k^2)\,\theta = 0 \qquad (5.1)$$

with boundary conditions (1.5).

The decrement and amplitude spectra are found by conventional means. The decrements are

$$\lambda_n = {}^1/_4\ (n+1)^2\pi^2 + k^2 + {}^1/_4\, p^2 \qquad (n = 0, 1, 2, 3, \ldots) \qquad (5.2)$$

Thus, all λ_n remain real at all flow velocities (monotonous perturbation decay), and increase quadratically with increasing p. We draw the attention to the difference between a thermal perturbation spectrum and the spectrum of hydrodynamic perturbations in one and the same flow [5]. In the latter all levels merge pairwise, and for large values of p oscillatory perturbations are only possible.

The amplitudes of perturbations, both even and odd with respect to x are

$$\theta_n = \exp\,({}^1/_2\, px)\ \cos\, {}^1/_2\ (n+1)\,\pi x \qquad (n = 0, 2, 4, \ldots)$$

$$\theta_n = \exp\,({}^1/_2 px)\ \sin\, {}^1/_2,\ (n+1)\,\pi x \qquad (n = 1, 3, 5, \ldots) \qquad (5.3)$$

A comparison of (5.3) with the amplitudes in a fluid at rest (2.6) shows that the perturbations are 'pressed' by the transverse flow against plane $x = 1$, and that with $p \gg 1$ these become localized in the boundary layer, the dimensionless thickness of which is of the order of ${}^1/p$.

It is interesting to note the validity of the decrement spectrum (5.2) for the case of thermally insulated boundaries, in which the absence of heat flow must be stipulated instead of conditions (1.5).

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